7.2　正規分布の平均値の区間推定の基

実数 α について $0 < \alpha < 1$ とする．確率変数 X は正規分布に従うとする． $\sigma[X] > 0$ とする． X を標準化した確率変数 $Z = \dfrac{X - \mathrm{E}[X]}{\sigma[X]}$ は標準正規分布 $\mathrm{N}(0,1)$ に従う．標準正規分布の累積分布関数を $\Phi(x)$ とおく：

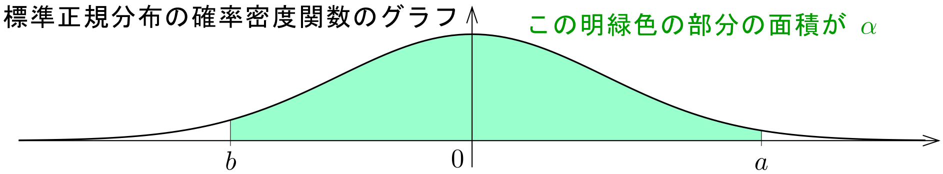
$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt \ .$$

各実数 a について，

$$\mathrm{P}(\, Z \leq a \,) = \Phi(a) \ ,$$

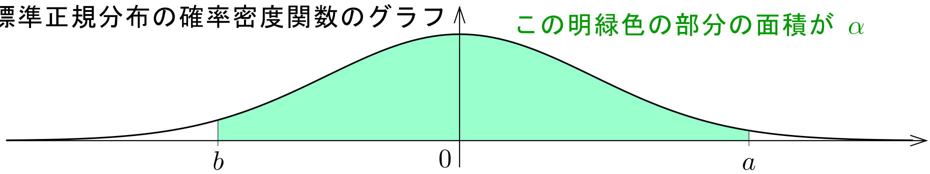
$$\Phi(-a) = 1 - \Phi(a) \ .$$

信頼度 α に対して次のような実数 a, b を考える： $\mathrm{P}(\,b \leq Z \leq a\,) = \alpha$.



信頼度 α に対して次のような実数 a, b を考える： $\mathrm{P}(\, b \leq Z \leq a\,) = \alpha$ ．

標準正規分布の確率密度関数のグラフ

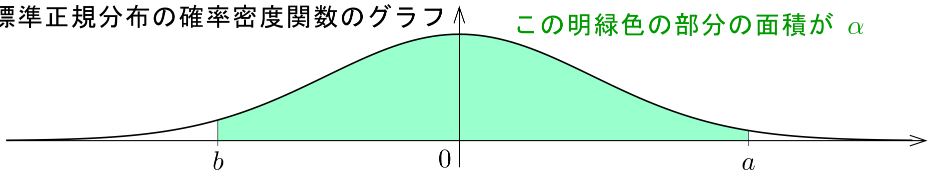


信頼度 α を満たす範囲で信頼区間をなるべく狭くするために，

$\mathrm{P}(\, b \leq Z \leq a\,) = \alpha$ である実数 a と b との間隔 $a - b$ をなるべく小さくする．

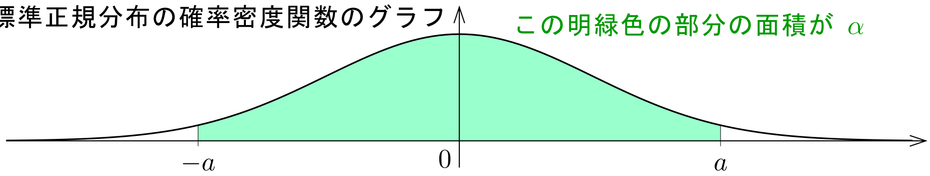
信頼度 α に対して次のような実数 a,b を考える： $\mathrm{P}(\, b \leq Z \leq a\,) = \alpha$ ．

標準正規分布の確率密度関数のグラフ



信頼度 α を満たす範囲で信頼区間をなるべく狭くするために，$\mathrm{P}(\, b \leq Z \leq a\,) = \alpha$ である実数 a と b との間隔 $a-b$ をなるべく小さくする．標準正規分布 $\mathrm{N}(0,1)$ の確率密度関数は偶関数なので， $b=-a$ とする．

標準正規分布の確率密度関数のグラフ



$\mathrm{P}(\, -a \leq Z \leq a\,) = \alpha$ である実数 a を求める．

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$$\begin{aligned} \mathrm{P}(\,-a \leq Z \leq a\,) &= \mathrm{P}(\,Z \leq a\,) - \mathrm{P}(\,Z \leq -a\,) \\ &= \Phi(a) - \Phi(-a) = \Phi(a) - \{1 - \Phi(a)\} \\ &= 2\Phi(a) - 1 \ . \end{aligned}$$

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よって

$$\begin{aligned} \mathrm{P}(\,-a \leq Z \leq a\,) = \alpha \iff 2\Phi(a) - 1 = \alpha \iff \Phi(a) = \frac{1+\alpha}{2} \\ \iff a = \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \ . \end{aligned}$$

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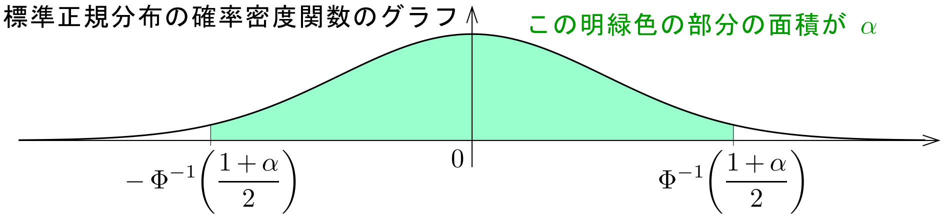
\end{aligned}$$

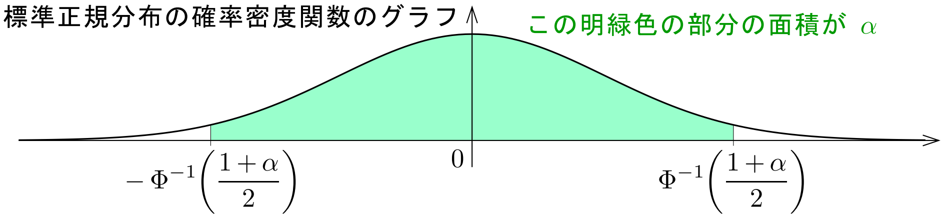
故に

$$\mathrm{P}\left(\,-\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq Z \leq \Phi^{-1}\left(\frac{1+\alpha}{2}\right)\,\right) = \alpha \ .$$

$$\mathrm{P}\left(\ -\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \le Z \le \Phi^{-1}\left(\frac{1+\alpha}{2}\right)\ \right)\ =\ \alpha\ .$$



$$\mathrm{P}\left(-\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq Z \leq \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \right) = \alpha \ .$$



$Z = \dfrac{X - \mathrm{E}[X]}{\sigma[X]}$ なので

$$\mathrm{P}\left(-\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq \frac{X - \mathrm{E}[X]}{\sigma[X]} \leq \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \right) = \alpha \ .$$

$$\mathrm{P}\left(\ -\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq \frac{X-\mathrm{E}[X]}{\sigma[X]} \leq \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \ \right) \ = \ \alpha \ \ .$$

$$\mathrm{P}\left(\, -\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq \frac{X-\mathrm{E}[X]}{\sigma[X]} \leq \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \, \right) \, = \, \alpha \ \ .$$

$\sigma[X] > 0$ なので,

$$\begin{aligned} & -\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq \frac{X-\mathrm{E}[X]}{\sigma[X]} \leq \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \\ \Longleftrightarrow \quad & -\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq X-\mathrm{E}[X] \leq \sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \end{aligned}$$

$$\mathrm{P}\left(-\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \le \frac{X-\mathrm{E}[X]}{\sigma[X]} \le \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \right) = \alpha \ .$$

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\iff \ & -\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \le X-\mathrm{E}[X] \le \sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \\
\iff \ & \sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \ge \mathrm{E}[X]-X \ge -\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right)
\end{aligned}$$

$$\mathrm{P}\left(-\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \le \frac{X-\mathrm{E}[X]}{\sigma[X]} \le \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \right) = \alpha \ .$$

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\iff \ & \sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \geq \mathrm{E}[X]-X \geq -\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \\
\iff \ & -\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq \mathrm{E}[X]-X \leq \sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \\
\iff \ & X-\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq \mathrm{E}[X] \leq X+\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \ .
\end{aligned}$$

$$\mathrm{P}\left(-\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq \frac{X-\mathrm{E}[X]}{\sigma[X]} \leq \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \right) = \alpha \ .$$

$\sigma[X] > 0$ なので,

$$\begin{aligned}
& -\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq \frac{X-\mathrm{E}[X]}{\sigma[X]} \leq \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \\
\iff & -\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq X-\mathrm{E}[X] \leq \sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \\
\iff & \sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \geq \mathrm{E}[X]-X \geq -\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \\
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\iff & X-\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq \mathrm{E}[X] \leq X+\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \ .
\end{aligned}$$

よって

$$\mathrm{P}\left(X-\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \leq \mathrm{E}[X] \leq X+\sigma[X]\,\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \right) = \alpha \ .$$

[定理 **7.2**] 標準正規分布 $\mathrm{N}(0,1)$ の累積分布関数を $\Phi(x)$ とおく．実数 α について $0<\alpha<1$ とする．正規分布に従う確率変数 X について，

$$\mathrm{P}\left(X-\sigma[X]\,\Phi^{-1}\!\left(\frac{1+\alpha}{2}\right) \leq \mathrm{E}[X] \leq X+\sigma[X]\,\Phi^{-1}\!\left(\frac{1+\alpha}{2}\right) \right) = \alpha \ .$$